Combinatorics

So far, we've come up with a very simple formula for calculating probabilities: ______
Unfortunately, calculating $N(E)$ and $N(\Omega)$ can be quite tricky, because there are often a lot more elements in those sets than we first realize. Combinatorics is the field of mathematics used to count the number of objects in a set. To realize why this is such a tricky area, determine what the elements belong to the sample space of the following random experiments - write down a few of the elements. Is it obvious (without any math) how many elements are in the sample space?

1. Draw one card from a standard deck of playing cards.

2. Roll three dice.

3. Draw a five-card poker hand from a standard deck of playing cards.

4. Randomly arrange seven girls in a line.

5. Guess which of 10 friends will show up at a party.

As these example should demonstrate, "counting" in this sense can be very difficult. The trick is to break these big random experiments down into a series of smaller random experiments, so that rather than worry about one big problem, we solve a series of smaller ones.

To start, there are two critical parts of combinatorics terminology that need to be discussed.

Order

Using regular English, when we say "order doesn't matter," we mean that we don't really care what the order actually is. To use some examples from the previous page, we probably don't care which order the girls are actually in. We probably don't care which of three dice roll a 6, nor do we care about the order of the cards we draw when we get our 5-card hand. We care more about how many dice roll a six, or which cards are in the hand. In this section, we're going to look at order in a different way, so be prepared.

In combinatorics, if you're asked whether or not order matters, what we really want to do is ask whether changing the order produces a something new that must be counted separately. For example, lets suppose that the seven girls are in the following order: Amanda, Lisa, Jamie, Dana, Karla, Susan, and Lily. By switching Lisa with Dana, the girls are in a different order. Since we're counting orderings, it's fairly obvious that this needs to be counted separately.

In contrast, consider drawing the 5-card hand 7H-QD-5H-10S-5C. If instead you were to draw 7H-QD-10S-5C-5H, this isn't going to change the hand. All that matters here is what cards you draw, not the order in which you drew them.

Finally, consider rolling three dice. Just as with two dice, even if we don't care which dice actually rolls each number, the number of ways to get each possible outcome (say 4-5-2) is determined by how many ways the individual dice can produce those outcomes. Thus, in combinatorics, order does matter for dice.

Replacement

A lot of the sets we're going to talk about are created by combining elements of simpler sets. For example, we create the set of possible rolls of 3 dice by combining the elements from the set of possible results in one roll. There are two ways that this combining can occur:

Without replacement: In sampling without replacement, elements used earlier in the series of mini-experiments are no longer available for subsequent mini-experiments. The most common example is drawing cards from a deck. This means that the sample from which the cards are drawn from changes every time (52, then 51, etc.), and the same card cannot be included twice. This reduces the number of possible hands that can be created.

With Replacement: In sampling with replacement, elements appearing earlier in the series of mini-experiments are still available for subsequent mini-experiments. This means that cards drawn earlier are replaced back into the deck. We can also use this for dice and coins, where the elements appearing earlier cannot disappear later.

Examples

1. Does order matter in the following experiments?

    (a) Choose 3 people to play 3 parts (Frank, Chloe, and a young child) in a play.

(b) Choose 4 people to sing to chant during a play.

2. Do the following experiments involve sampling with or without replacement?

   (a) Rearrange the letters from the word "careful".

   (b) Create a password containing exactly 8 lowercase letters.

Basic Counting Rule (BCR)

Idea

Suppose we have a random experiment that consists of doing the same thing r times - for example, rolling 6 dice, or flipping a coin 10 times. We can instead treat this as a series of r experiments. If each of these smaller experiments has m possible outcomes, then we can calculate the total number of possible outcomes by:

Order

The BCR is used when order matters. Each different ordering must be counted separately.

Sampling

The BCR is used when you sample with replacement.

Examples

1. Suppose you have 5 dice. Die A has 6 sides with sample space $\{1, 2, \ldots 6\}$. Die B has 10 sides with sample space $\{1, 2, \ldots, 10\}$. Die C has 4 sides, $\{2, 4, 6, 8\}$. Die D has 6 sides, $\{3, 6, 9, 12, 15, 18\}. Finally,$Die E has 4 sides: $\{A, B, C, D\}$.

   (a) Suppose you roll dice A,B,C, and D. How many possible outcomes can you have?

(b) Suppose you roll die E. The dice corresponding to the three sides showing are then rolled. How many possible outcomes are possible for the sum of these dice? Why is this not an example of a BCR problem? Can you solve it anyways?

(c) Let S be the sum of the 4 numeric dice, and let $E = \{s : s \in S, s \text{ even}\}$, be the event that the the sum of the 4 numeric dice is even. What is $N(E)$?

2. Suppose Indiana license plates consist of 2 numeric digits (not including 00) followed by one letter, then three digits that might be letters or numbers. How many different license plates are there?

3. The Birthday Problem: What is the probability that no two students in this class share the same birthday?

Permutation Rule

Permutations

A permutation is an ____________________ arrangement of objects.

Examples

- Words. "Ten" is not the same as "net"; "three" is different from "there".
- Parts in a play. If there are two parts in a play, that of a young girl and of an old man, and there are three actors to choose from - a young girl, an old man, and a middle aged woman, matching the parts to the appropriate actors makes a big difference.
- Imagine trying to randomly assign 4 different lollipop flavors to 4 children. If you want 4 happy kids, it's best to make sure that each kid likes the flavor they get (if possible).

Versions

The permutation rule actually has two different versions, depending on whether duplicate objects exist. Duplicate items are considered to be identical, so that switching two duplicate items does not constitute a different arrangement. For example, switching the two 'e's' in the word "three" does not change the word in any way. All copies of the letter 'e' are considered identical.

No Duplicate Items

Conditions:

Action: Permute. The permutation operator is written two different ways:

Special Case: When $m = r$, then

In Words: Suppose we m different objects, and we are taking out r of these objects, one at a time. The first object we take can be any of the m objects, so we have m choices in the first random experiment. No matter what the first object is, we have $m - 1$ choices for the second object. We can continue this pattern until we have selected all r objects. Although the *choices* depend on which objects have been taken already, the *number* of choices for the i^{th} random experiment is always the same! This means that we can use the BCR. Consequently, the total number of combinations is:

Duplicate Items

Conditions: The Permutation rule for duplicate items is applied when we are arranging a collection of m objects, which consists of $m_1, \ldots, m_n$ copies of n distinct objects.
Note 1: $m = m_1 + \ldots + m_n$
Note 2: This permutation rule always assumes that all the objects are to be arranged, not just a subset of r of them.

Action: Divide the total number of permutations by the ____________________ of the number of repeats:

In words: Start by calculating the total number of permutations assuming all items are different. (Artificially numbering the duplicates is one way to do this: 'e_1', 'e_2', etc.) Now, we want to remove all permutations in which *only* the duplicate item changes. To do this, take any permutation of all the items, and examine how many ways the duplicate items can be permuted. For example, consider the word "sassy" as a permutation of letters. Note that we have 3 's's', so let's rewrite the word as "$s_1as_2s_3y$".

Ignoring the 'a' and the 'y', the 3 's's' can be permuted in _______________ ways. Now, this means that every pattern of 'a', 'y', and the 's's will be duplicated

_______________ times, because we always can always permute the 's's' that many times. Consequently, the final answer is

Examples

1. How many ways can 7 kids be ordered in a line?

2. If the kids above are randomly placed in an ordering, what is the probability that they are in alphabetical order, assuming that three children are named Michael?

3. A theater is holding auditions for a play that consists of 10 roles. If there are no restrictions on who can apply for each part, and 20 people show up for the audition, how many ways can the theater choose the cast?

4. Consider the letters that make up the word STATISTICS.
   (a) How many ways can these letters be reordered?
   (b) What is the probability that a permutation taken at random has all three 'T's located together?
   (c) What is the probability that there is an 'S' on at least one of the ends?

Combination Rule

Combinations

A combination is an ____________________ arrangement of objects.

Examples

- A Committee
- A group of singers in Choir. (contrast with parts in a play)
- A hand of playing cards.

Conditions

The Combination rule is used when we are taking a set of r objects from a collection of m different objects ________________________ them.

Action

The combination operator is written two different ways:

In Words

Start by calculating the number of permutations possible when selecting r objects from a collection of m objects. The idea here is almost identical to that of calculating the number of permutations possible when some objects are repeated. We want eliminate the duplication caused by listing all different permutations. The number of permutations of r objects is __________, so the formula must be:

Examples

1. The US senate consists of 100 senators, two from each state. A 5 person committee is to be chosen from these senators.

   (a) If there are no restrictions, how many ways can a 5 person committee be chosen?

   (b) Suppose that only one senator from each state is allowed. How many ways can the committee be chosen now?

2. How many ways can a group of 10 children be divided into two teams of five children?

3. The Indiana Powerball lottery consists of drawing 5 numbers from $\{1, 2, \ldots, 55\}$ and one number (the POWERBALL) from the numbers $\{1, 2, \ldots, 42\}$. To win, all 6 numbers must match, but the order of the 5 regular numbers is not important.
   (a) What is the probability of winning the Indiana Powerball?

   (b) What is the probability of matching all 5 regular numbers, but not the Powerball?

Complex Problems

The rules from the previous section are often insufficient by themselves to solve many problems in combinatorics. There are many questions that can be asked that require combining several different rules at once, or using other tricks. While not all-inclusive, here are some general principles to help.

- Try to simplify complex problems wherever possible. Try to see if you can separate out smaller problems that the above rules can be applied to.
- Look for mutually exclusive events. Mutually exclusive events are always ____________.
- Sometimes subsets of the same population are used to create both permutations and combinations. Make sure to determine whether or not the same object can exist in both the permutation and the combination. If they can, how would you combine the smaller permutation problem and the combination problem?

  If not, then what would you do?
- Once you have simplified a complex problem into smaller parts, use the same rules to combine the parts.

For example:

1. Suppose you play 5 games of poker. In the first 3 games, you draw a hand of 5 cards from a standard 52 card deck. In the last two games, you draw a hand of 7 cards. How many combinations of hands and cards can you have?

2. Suppose a teacher has a class of 30 students. She'd like to choose a class president, vice president, and a council of 4 children. The same student cannot hold more than one position. How many ways can she fill the positions?

3. A password is to consist of 7 or 8 alphanumeric digits. The same number or letter may not be repeated more than once.

4. A director is casting a play that requires 2 children, 4 men, and 5 women, and a "choir" of 4 men and 4 women. If 42 children, 18 men, and 23 women come to the auditions, and any adult who is not cast for a main part is considered for the choir, how many ways can the director fill all the roles?

5. Using the previous problem, what is the probability that David is part of the play, if David is an adult male?

6. Additional practice: Calculate the probability of getting various poker hands. Poker hands consist of 5 cards from a standard deck of playing cards. Order does not matter. Answers to these problems are available at `http://en.wikipedia.org/wiki/Poker_probability`,but I would strongly recommend trying to work them out yourself, and using the web only to check your work.

   (a) A straight flush consists of 5 cards in the same suit in numerical order:

   $$5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit$$

   The Ace can be at the top or the bottom card, but it cannot "wrap around". I.e. $Q\heartsuit, K\heartsuit, A\heartsuit, 2\heartsuit, 3\heartsuit$ is *not* a straight flush.

   (b) Four of a kind consists of 4 cards of the same rank. There are no restrictions on the 5^{th} card:

   $$7\heartsuit, 7\spadesuit, 7\diamondsuit, 7\clubsuit, Q\heartsuit$$

   (c) A full house consists of 3 cards of one rank, and 2 cards of a second rank:

   $$7\heartsuit, 7\spadesuit, 7\diamondsuit, 3\heartsuit, 3\clubsuit$$

(d) A flush consists of 5 cards of the same suit. Generally, a straight flush is not counted as a flush.

$$8\spadesuit, A\spadesuit, Q\spadesuit, 9\spadesuit, 4\spadesuit$$

(e) A straight consists of 5 cards in numerical order. As with a straight flush, Aces cannot wrap around. A straight flush is generally not counted as a straight.

$$8\clubsuit, 9\heartsuit, 10\diamondsuit, J\heartsuit, Q\clubsuit$$

(f) Three of a kind consists of 3 cards of the same rank. The other two cards cannot have the same rank.

$$7\heartsuit, 7\spadesuit, 7\diamondsuit, Q\heartsuit, 3\clubsuit$$

(g) Two pair consists of 2 cards of one rank, 2 cards of a second rank, and a fifth card that does not match either of the other two ranks.

$$7\heartsuit, 7\spadesuit, 9\diamondsuit, 3\heartsuit, 3\clubsuit$$

(h) A pair consists of 2 cards of one rank. The other three cards cannot have the same rank.

$$A\clubsuit, K\spadesuit, 6\diamondsuit, 3\diamondsuit, 3\clubsuit$$