

Counting Poker Hands*

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Disclaimer: This handout is in no way promoting gambling or play cards, but simply uses this model because it is combinatorially interesting.

Poker, aka Earth Poker, is played with a 52 card deck. Each card has two attributes: rank and suit. The *rank* of a card can be $\{2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A\}$, giving 13 possible ranks. The *suit* of a card can be $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$, giving 4 possible suits. Jokers and wildcards complicate this picture considerably and will be considered separately once the basics have been covered.

The purpose of this handout is to help clarify the problem of counting variations of poker hands. To this end, several detailed examples are given below of the classic situations. Variations of the game are also given both as examples and as exercises to do on your own. None of these exercises is required, but I recommend that you try a few as a way to study for the final exam. Solutions will not be posted. Look to the examples for guidance, and ask me if you have any questions.

Example: 5 card stud.

Let's start by playing a 5 card game. The order in which the cards are dealt is irrelevant. We want to rank the types of hands that a player can get. In the table below, the hands are ranked by how common they are, beginning with the least common. Below the table is an explanation for each row, including a description of the hand and a detailed combinatorial proof for how to count how many such hands can occur. To begin, notice that there are exactly

$$\binom{52}{5} = 2,598,960$$

possible 5 card hands.

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<i>hand</i>	<i>how to count</i>	<i>number</i>
straight flush	$\binom{10}{1} \cdot \binom{4}{1} =$	40
four of a kind	$\binom{13}{1} \cdot \binom{52-4}{1} =$	624
full house	$\binom{13}{1} \cdot \binom{4}{3} \cdot \binom{12}{1} \cdot \binom{4}{2} =$	3744
flush	$\binom{13}{5} \cdot \binom{4}{1} - 40 =$	5108
straight	$\binom{10}{1} \cdot \binom{4}{1}^5 - 40 =$	10200
three of a kind	$\binom{13}{1} \cdot \binom{4}{3} \cdot \binom{12}{2} \cdot \binom{4}{1}^2 =$	54912
two pair	$\binom{13}{2} \cdot \binom{4}{2}^2 \cdot \binom{52-8}{1} =$	123552
one pair	$\binom{13}{1} \cdot \binom{4}{2} \cdot \binom{12}{3} \cdot \binom{4}{1}^3 =$	1098240

For each of the computations, we can choose the rank and suit of each card. Sometimes, this isn't even necessary, as with picking an arbitrary 5 card hand. For consistency, we will always pick the rank first, though sometimes it may be more natural to pick the suit first.

- A *straight flush* is a hand with 5 consecutive ranks, all of which are the same suit. For the purposes of straights, an *A* may be high ($K+1$) or low (1). To choose a straight, just pick the lowest card of the straight, then you must take the next four cards of that same suit. Choose the rank of the lowest card in $\binom{10}{1}$ ways (remember that you must have at least 4 ranks above the lowest card!), and choose the suit of the card in $\binom{4}{1}$ ways.
- *Four of a kind* consists of all four cards of some rank, and a fifth card, necessarily of a different rank. Since the choice for the fifth card depends on the four matching cards, choose it last. To choose the four matching cards, there are $\binom{13}{1}$ possible ranks, and you must take all the cards of that rank. For the fifth card, just take any of the remaining cards in $\binom{52-4}{1}$ ways.
- A *full house* is three matching ranks and two other matching ranks. To choose the three of a kind, pick any rank in $\binom{13}{1}$ ways and choose 3 suits for that rank in $\binom{4}{3}$ ways. To pick the pair, choose any of the remaining ranks in $\binom{12}{1}$ ways and any two suits in $\binom{4}{2}$ ways.
- A *flush* is a hand of all the same suit, but which is *not* also a straight flush (because this is a better hand). To choose a flush, first choose any 5 different ranks in $\binom{13}{5}$ ways, and then choose a suit in $\binom{4}{1}$ ways. Finally, you must subtract the number of straight flushes, which we computed above to be 40.
- A *straight* is a hand of consecutive ranks, but which is *not* also a straight flush (again, this would be a better hand). To choose a straight, pick the lowest rank in $\binom{10}{1}$ ways (remember that an *A* can be high or low!) and then choose a suit for each card (independently) in $\binom{4}{1}^5$ ways. Finally, don't forget to subtract off the number of straight flushes.

- *Three of a kind* is a hand with three cards of the same rank, and two cards of two different ranks. Be careful not to count in full houses! To choose the three of a kind, pick a rank in $\binom{13}{1}$ ways, then choose three different suits in $\binom{4}{3}$ ways. For the last two cards, make sure they don't pair. To do this, choose two different ranks of the remaining ranks in $\binom{12}{2}$ ways, then choose a suit for each card (independently) in $\binom{4}{1}^2$ ways.
- *Two pair* is a hand with two pairs of different ranks and a third different rank. To avoid picking the hand in order, choose the two pair at once. Pick two different ranks in $\binom{13}{2}$ ways, and pick suits for each pair (the pairs are independent) in $\binom{4}{2}^2$ ways. Finally, pick the last card to be of a different rank from the pairs. This is most easily done by throwing out all cards of those two ranks and picking any of the remaining cards in $\binom{52-8}{1}$ ways.
- *One pair* is a hand with one and only one pair in it. To choose the pair, pick a rank in $\binom{13}{1}$ ways and pick two different suits in $\binom{4}{2}$ ways. For the remaining three cards, they must not pair with each other or with the original pair. To ensure this, choose three different ranks other than the rank of the pair in $\binom{12}{3}$ ways. Then choose a suit for each card (independently) in $\binom{4}{1}^3$ ways.

Questions:

1. When counting how many ways one could get three of a kind, why was the solution not just $13 \cdot \binom{4}{3} \cdot \binom{52-4}{2}$? By how much does this overcount?
2. For one pair, why is the last term $\binom{4}{1}^3$ instead of $\binom{4}{3}$?
3. Use the values of the table to determine how many ways there are to get one pair or three of a kind (or both).
4. Using any or all of the information in the table, find a quick way to determine the number of 5 card hands with nothing.

Exercise: 3 card stud.

For this game, each player is dealt three cards. Complete the computations for the table below, and give a combinatorial argument for your solution. The numbers provided should be a good check to whether your answer is correct.

<i>hand</i>	<i>how to count</i>	<i>number</i>
straight flush	=	48
three of a kind	=	52
straight	=	720
flush	=	1096
one pair	=	3744

Questions:

1. Are you more likely to get a flush in 3 card stud or three of a kind in 5 card stud? Determine the probability of each and compare.
2. In 5 card stud, a straight is more likely than a flush. However, in 3 card stud, a flush is more likely than a straight. Try to justify why one should expect that.
3. Determine the probability for the following event: You are dealt a pair in a 5 card poker hand. Then you trade in your three unmatched cards for three new cards (you may assume from a different deck) and then you have a full house!

Example/Exercise: 7 card stud.

The number of cards has changed, but the idea remains the same. Fill in the missing entries in the table below. The numbers should help you determine if your solution is correct. If a solution is given, explain why it is correct. Some terminology: *overcrowded house* means 4 of a kind and 3 of another kind; *crowded house* means 3 of a kind and two pair.

<i>hand</i>	<i>how to count</i>	<i>number</i>
overcrowded house		= 624
two threes-of-a-kind	$\binom{13}{2} \cdot \binom{4}{3}^2 \cdot \binom{52-8}{1}$	= 54912
crowded house		= 123552
three pair		= 2471040
full house	$\binom{13}{1} \cdot \binom{4}{3} \cdot \binom{12}{1} \cdot \binom{4}{2} \cdot \binom{11}{2} \cdot \binom{4}{1}^2$	= 3294720
three of a kind		= 6589440
two pair	$\binom{13}{2} \cdot \binom{4}{2}^2 \cdot \binom{13-2}{3} \cdot \binom{4}{1}^3$	= 29652480
one pair		= 63258624

Example: Two Pair in 7 card stud.

The general rule is that sets of the same size need to be chosen together. For two pair, there are two set sizes: one (for the single unpaired cards) and two (for the two pairs).

Choose all of the pairs at once, by choosing two different ranks, $\binom{13}{2}$, and then choosing the suits for each pair, $\binom{4}{2}^2$. The suits for a pair must be chosen together without repetition, which gives the term $\binom{4}{2}$. Since the two different pairs are independent as far as suits are concerned, the term is squared.

To choose the single cards, they must not match up with either of the ranks of the two pair. They must also be distinct from one another. Therefore choose the ranks without repetition from the remaining ranks, and do so at one time, giving the term $\binom{13-2}{3}$. Since the suit of each of these cards is independent, each card contributes one factor of $\binom{4}{1}$, giving a total of $\binom{4}{1}^3$.

French Revolution: This is a variant of 5 card stud, in which it is bad to get royals: Jacks, Queens and Kings. The ranking of hands is as follows: best poker hand with no royals; if no such hand exists, then best poker hand with one royal; and so on.

1. What is the best possible hand? Determine the probability of getting this hand.
2. How many hands are there with 3 of a kind and only one royal?
3. What is the probability of getting one pair and no royals? How does this compare with hands in regular 5 card stud?
4. A similar variant of this game is called Jackass. It has the same ranking, but if you are dealt the Jack of Diamonds and no other royals, then you win. What is the probability that this happens?

Generalized Poker: An (r, s) -deck of cards has r different ranks and s different suits, for a total of rs cards in the deck. Suppose you are dealt cards from the deck, where the order in which the cards are dealt doesn't matter. Determine the number of possible hands of the following types.

1. In a 5 card hand, getting a full house.
2. In a 9 card hand, getting three pair.
3. In a 4 card hand, getting three of a kind.
4. In a 20 card hand, getting four of a kind, two threes-of-a-kind and three pair.