

Mock Exam for Final Discrete Mathematical Structures CS3233

December, 2005

The final exam is cumulative. This mock exam covers material starting with that presented in lecture 11/07/2005, which was not previously tested. This includes material in sections 3.4, 3.5, 4.1-3, and 5.1-3. Students should refer to midterms 1 and 2, and to the corresponding mock exams and homeworks, for assistance in studying material previously tested.

1. Determine whether the following are valid recursive definitions of a function $f : N \rightarrow Z$:

- (a) Valid or invalid: $f(0) = 0, f(1) = 1, f(n) = 2f(n-2)$ for $n > 1$
- (b) Valid or invalid: $f(0) = 0, f(1) = 1, f(n) = 2f(n)$ for $n > 1$
- (c) Valid or invalid: $f(0) = 0, f(1) = 1, f(n) = 2f(n+1) + f(n+2)$ for $n > 1$

2. Let the set of positive propositional formulas Pos be recursively defined as follows:

Basis: $p \in Pos$ for all propositional variables p .

Recursive step: if $\varphi \in Pos$ and $\psi \in Pos$, then $\varphi \wedge \psi \in Pos$, $\varphi \vee \psi \in Pos$, and $(\varphi) \in Pos$.

Given truth assignments σ_1 and σ_2 , we write $\sigma_1 \preceq \sigma_2$ if σ_2 makes true every variable that σ_1 makes true (and possibly some others)

Prove the following by structural induction: Consider any $\varphi \in Pos$ and any truth assignments σ_1 and σ_2 such that $\sigma_1 \preceq \sigma_2$. If σ_1 satisfies φ , then so does σ_2 .

3. What is the best big-O estimate of the worst-case complexity of mergesort? Justify your answer.

4. In this section of the exam, please do not fully evaluate your expressions. Simplify your answers so that they are expressed in terms of factorials and exponents, but not in terms of $C(n, r)$ or $P(n, r)$.

- (a) How many r -letter words can be constructed using an alphabet of size n ?
- (b) How many ways are there to order a set of size n ?
- (c) How many ways are there to order a set of size n if the first r places are always occupied by the same r elements?
- (d) How many ways are there to partition a set of size n into two sets, one of which has size r ? (A *partition* of set S is a collection of subsets of S such that the intersection of any two sets in the collection is empty and the union of all the sets in the collection is the whole original set.)
- (e) How many ways are there to partition a set of size n into two sets A and B ? Note that it matters which set is called A and which one is called B .
- (f) How many 8-bit sequences have exactly 3 0's?
- (g) How many ways are there to put 6 keys on a circular key ring? The two sides of the ring are the same and the two sides of the keys are the same. (So both ring and keys are symmetrical.) There is also no way to tell which key is first.
- (h) What is the probability of being dealt a particular hand of 5 cards from a deck of 52 distinct cards?
- (i) What is the conditional probability of being dealt a particular hand of 5 cards from a deck of 52 distinct cards given that another player is also dealt a hand of 5 cards?
- (j) Suppose a permutation of all 26 lowercase letters is selected at random with each permutation equally likely.

- i. What is the probability that the first 13 letters of the permutation are in order?
 - ii. What is the probability that a and b are not next to each other?
- (k) Given a 6-sided die for which each of the values 1 through 6 comes up with equal probability, what is the conditional probability of rolling a 6 given that the last 3 times you rolled you also got a 6.
- (l) You and Humayun are playing poker, a game which he claims not to understand. You are each simultaneously and fairly dealt hands of 5 cards from the same 52-card deck. There are only 4 ways of getting a royal flush and these 4 hands do not share any cards. What is the conditional probability of your being dealt a royal flush given that Humayun is also dealt a royal flush?
- (m) What is the expected value of a roll of one 6-sided die assuming each of the values 1 through 6 comes up with equal probability?
- (n) Given a 6-sided die that is weighted so that each of the values 2 through 6 comes up with equal probability, but 1 comes up twice as often as the other values, what is the probability distribution for a roll of this die? (Give the probability of getting each one of the 6 values.)
- (o) What is the expected value of a roll of one 6-sided die weighted as described in the previous problem?