Math 352 — Fall 2006

Test 1

Solution Key

1. Suppose that you've just looked on the national weather service site, and seen that the probability of rain on Saturday is 60%, and the probability of rain on Sunday is 70%. Treating the two days as separate experiments, what is possible range for the probability of rain over the weekend? [8 points]

   Denoting "rain on Saturday" by A and "rain on Sunday" by B, the question asks for the range of possible values for $\mathsf{P}(A \cup B)$. Since $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B)$, and the last term is between 0 and 1, a really loose set of bounds would be

   $$0.3 = 0.6 + 0.7 - 1 \leq \mathsf{P}(A \cup B) \leq 0.6 + 0.7 = 1.3$$

   (this was Olofsson 1.10 from PS 1 with numbers plugged in) Of course, probabilities cannot exceed 1, and $\max(\mathsf{P}(A), \mathsf{P}(B)) \leq \mathsf{P}(A \cup B)$, so better bounds would be

   $$0.7 \leq \mathsf{P}(A \cup B) \leq 1.$$

2. Oskar and Maria are talking. Oskar says that independence means that two events have nothing to do with each other, so since disjoint events have nothing to do with each other, they must also be independent. Is Oskar's statement always correct, sometimes correct, or never correct? Justify your answer carefully. [6 points]

   Oskar is very rarely correct. If two events are disjoint, then $A \cap B = \emptyset$ whereas two events are independent if and only if $\mathsf{P}(A \cap B) = \mathsf{P}(A) \times \mathsf{P}(B)$. So... Oskar will be correct if and only if $\mathsf{P}(A) = 0$ or $\mathsf{P}(B) = 0$. (This is Example 1.5.10 on page 37 of the text. It is also simple enough to do by writing down definitions of "disjoint" and "independent".)

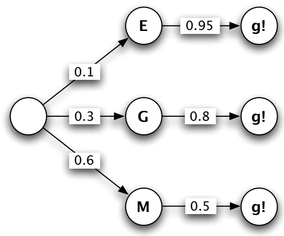
3. Oskar and Maria are talking probability. Oskar says that if $\mathsf{P}(A|B) = \mathsf{P}(B|A)$, then $\mathsf{P}(A) = \mathsf{P}(B)$. Maria disagrees entirely. Is Oskar's claim correct always, sometimes, or never? Justify your answer! [8 points]

   $$\begin{aligned}
   & & \mathsf{P}(A|B) &= \mathsf{P}(B|A) & & \\
   &\iff & \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(B)} &= \frac{\mathsf{P}(B \cap A)}{\mathsf{P}(A)} & & \text{by definition} \\
   &\iff & \mathsf{P}(A)\,\mathsf{P}(A \cap B) &= \mathsf{P}(B)\,\mathsf{P}(A \cap B) & & \text{cross multiplying} \\
   &\iff & [\mathsf{P}(A) - \mathsf{P}(B)]\,\mathsf{P}(A \cap B) &= 0 & & \text{gathering and factoring.}
   \end{aligned}$$

   This will hold if and only if at least one of the two factors is zero, which happens whenever either $\mathsf{P}(A) = \mathsf{P}(B)$ or $\mathsf{P}(A \cap B) = 0$. Thus Oskar's claim is right except when A and B are disjoint, when the probabilities can be totally unrelated. (This Other Problem 1 on PS 3.)

4. Johnny Overstatement announces football on ESPN18. As his name suggests, he tends to exaggerate how good each play is. If a play is excellent, he will call it "the greatest ever" 95% of the time. If a play is good, he will call it "the greatest ever" 80% of the time. If a play is mediocre, he still will call it "the greatest ever" 50% of the time. Oskar is an avid football fan, and has kept his own statistics. He knows that plays are mediocre, good, and excellent with probabilities 60%, 30%, and 10%, respectively. After 18 straight hours of watching football, Oskar stumbles upon the James Mason–Fenimore Cooper football game, which is being announced by Johnny O. As Oskar is getting some food from the fridge, he hears "the greatest ever" come booming from the TV. What is the chance that a running back ran for 1 yard (a completely mediocre play)? [15 points]

   Using capital letters for the actual type of the play, and "g!" for the play being called "the greatest ever", the question asks for $\mathsf{P}(M|g!)$. This can be done symbolically or with a (simplified) tree diagram:

   So

$$\mathsf{P}(M|g!) = \frac{0.6 \times 0.5}{0.6 \times 0.5 + 0.3 \times 0.8 + 0.1 \times 0.95} = \frac{0.3}{0.3 + 0.24 + 0.95} \approx 0.472.$$

   (This is like the Harablee-Badd-Syndrome example from class, and exercise 1.76 from Olofsson except there are 3 categories instead of 2.)

5. Frank and Anita are playing a game they invented called seven-11-doubles. One player is the shooter, and is in charge of rolling the dice. If the dice come up seven or 11, the shooter wins. If the dice come up doubles (both dice the same), the shooter loses. Otherwise, the game continues, and the shooter rolls the dice again. What is the probability that the shooter wins? [12 points]

   Just like the craps example in class, it is best to condition on the game ending. Define $W := \{\text{Shooter rolls 7} \cup \text{Shooter rolls 11}\}$ and $L := \{\text{Shooter rolls doubles}\}$ These are two disjoint events which determine the end of the game. They are marked in the table below

		Die 2					
		1	2	3	4	5	6
	1	L					W
	2		L			W	
Die 1	3			L	W		
	4			W	L		
	5		W			L	W
	6	W				W	L

   From all this,

$$\mathsf{P}(\text{shooter wins}) = \frac{8/36}{8/36 + 6/36} = 8/14 \approx 0.571$$

   Not a very fair game! (This is similar to the craps from class, except far simpler.)

6. Dean has gone to the church picnic with a big appetite.

   (a) He buys \$3 worth of dimes at the cake booth, and bets one dime per spin of the wheel on number 43. Make some reasonable assumptions about how the spins could be modelled and find the chance that he wins a cake at the cake booth, assuming there are 100 numbers on the wheel. Be sure to state your assumptions. [12 points]

   Start by assuming that the spins are independent of one another and that all the numbers are equally likely. If W_i denotes the event he wins on spin i, and C denotes the event he wins a cake, then $C = W_1 \cup W_2 \cup \ldots \cup W_{30}$, since he just needs to win at least once. Now, *or*s are hard to work with, but *and*s are much easier. So...

$$\begin{aligned}
\mathsf{P}(C^c) &= \mathsf{P}((W_1 \cup W_2 \cup \ldots \cup W_{30})^c) && \text{complementing both sides} \\
&= \mathsf{P}(W_1^c \cap W_2^c \cap \ldots \cap W_{30}^c) && \text{De Morgen's Laws} \\
&= \left(\frac{99}{100}\right)^{30} && \text{independence} \\
&\approx 0.740
\end{aligned}$$

   Thus, he has roughly a $1 - 0.74 = 0.26$ chance of winning a cake. (This is like the example where Oskar is looking for a job from class.)

   (b) After leaving the cake booth, he plays blackjack in the hopes of winning enough money to pay for his \$8 chicken dinner. Unfortunately for him, the probability that the house wins is 54% for each hand. The hands are independent, because a new deck is used each time. If he arrives at the blackjack table with \$40 of betting money in his pocket, and bets \$2 each game, what is the chance that he'll win enough money for dinner? [15 points]

   This is a simple gambler's ruin problem with one twist: the chips are worth \$2, so Dean needs to win 4 chips while he is willing to risk 20. From the problem statement, $q = 0.54$, so $p = 0.46$.

$$\mathsf{P}(\text{wins } 4) = \frac{1 - \left(\frac{0.54}{0.46}\right)^{20}}{1 - \left(\frac{0.54}{0.46}\right)^{20+4}} \approx 0.516.$$

   (This is a gambler's ruin problem like Olofsson 1.98 and Other Problem 1 on PS 4.)

7. A transmitter sends 0's and 1's to a receiver. Both digits are equally likely to be corrupted: there is a 5% chance for each bit to be reversed between the transmitter and the receiver. Suppose that three times as many 0's are sent as 1's. [problem worth a total of 24 points]

   (a) If the transmitter sends the sequence 01, what is the probability that the receiver hears 01?

   The probability that the receiver hears what is sent is simply

$$\mathsf{P}(\text{first bit OK} \cap \text{second bit OK}) = 0.95^2 = 0.9025,$$

   assuming independence of errors.

   (b) If the receiver hears the sequence 01, what is the probability that 01 was sent?

   Using the notation $A := 0$ sent, $B := 1$ sent, $a := 0$ heard, and $b := 1$ heard, the problem asks for $\mathsf{P}(AB|ab)$. This can be found using Bayes Rule.

$$\begin{aligned}
\mathsf{P}(AB|ab) &= \frac{\mathsf{P}(AB \cap ab)}{\mathsf{P}(ab)} \\
&= \frac{\mathsf{P}(ab|AB)\,\mathsf{P}(AB)}{\mathsf{P}(AB)} \\
&= \frac{\mathsf{P}(ab|AB)\,\mathsf{P}(AB)}{\mathsf{P}(ab|AB)\,\mathsf{P}(AB) + \mathsf{P}(ab|AA)\,\mathsf{P}(AA) + \mathsf{P}(ab|BB)\,\mathsf{P}(BB) + \mathsf{P}(ab|BA)\,\mathsf{P}(BA)} \\
&= \frac{0.95^2 \times 0.75 \times 0.25}{0.95^2 \times 0.75 \times 0.25 + 0.95 \times 0.05 \times 0.75^2 + 0.05 \times 0.95 \times 0.25^2 + 0.05^2 * 0.25 \times 0.75} \\
&\approx 0.849
\end{aligned}$$

   It can also be found by finding the probabilities of hearing properly for each digit, and then multiplying:

$$\begin{aligned}
\mathsf{P}(A|a) &= \frac{\mathsf{P}(a|A)\,\mathsf{P}(A)}{\mathsf{P}(a|A)\,\mathsf{P}(A) + \mathsf{P}(a|B)\,\mathsf{P}(B)} && \text{Bayes Rule} \\
&= \frac{0.95 \times 0.75}{0.95 \times 0.75 + 0.05 \times 0.25} = \frac{57}{58} && \text{plugging in} \\
\mathsf{P}(B|b) &= \frac{\mathsf{P}(b|B)\,\mathsf{P}(B)}{\mathsf{P}(b|B)\,\mathsf{P}(B) + \mathsf{P}(b|A)\,\mathsf{P}(A)} && \text{Bayes Rule} \\
&= \frac{0.95 \times 0.25}{0.95 \times 0.25 + 0.05 \times 0.75} = \frac{19}{22} && \text{plugging in}
\end{aligned}$$

   Thus, $\mathsf{P}(AB|ab) = \mathsf{P}(A|a)\,\mathsf{P}(B|b) = \frac{57}{58} \times \frac{19}{22} \approx 0.849$ (Both parts of this are the same as the two parts of exercise 1.81 in Olofsson, except with different probabilities of success.)